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9th Annual Meeting of the Bulgarian Section of SIAM
December 18-19, 2014
Sofia

BGSIAM'14

PROCEEDINGS

HOSTED BY THE INSTITUTE OF MATHEMATICS AND INFORMATICS
BULGARIAN ACADEMY OF SCIENCES

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PREFACE

The 9th Annual Meeting of Bulgarian Section of SIAM (BGSIAM) took part on December 18 and 19, 2014 and was hosted by the Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia. The conference support provided by SIAM, as the major international organization for Industrial and Applied Mathematics, is very highly appreciated.

During the 9th Annual Meeting of BGSIAM (BGSIAM'14) a wide range of problems concerning recent achievements in the field of industrial and applied mathematics were presented such as: Numerical Methods and Algorithms; Control Systems and Applications; Partial Differential Equations and Applications; Neurosciences (Neural Networks); Equations of Mathematical Physics, etc. Following the established tradition, the conference provided a forum for exchange of ideas between scientists, who develop and study mathematical methods and algorithms, and researchers, who apply them for solving real life problems.

More than 50 participants from universities, institutes of the Bulgarian Academy of Sciences and also from outside the traditional academic departments took part in BGSIAM'14. They represent most of the strongest Bulgarian research groups in the field of industrial and applied mathematics. We are very glad to report that young researchers, Post Docs and Ph.D.students took part during BGSIAM'14. Organization of special sessions for young researchers is the main goal of BGSIAM in our future conferences.

Founded on January 18, 2007, the Bulgarian Section of SIAM (http://www.math.bas.bg/IMIdocs/BGSIAM/index.html) was officially approved by the SIAM Board of Trustees on July 15, 2007. The activities of BGSIAM follow the general objectives of SIAM, as established in its Certificate of Incorporation: to further the application of mathematics to industry and science; to promote basic research in mathematics leading to new methods and techniques useful to industry and science; to provide media for the exchange of information and ideas between mathematicians and other technical and scientific personnel. The role of SIAM is very important for promotion of interdisciplinary collaboration between applied mathematics and science, engineering and technology in the Republic of Bulgaria.

LIST OF INVITED LECTURES:

- VASSIL ALEXANDROV
  Barcelona Supercomputer Center,
  Department of Computer Science, Spain
  TOWARDS NOVEL SCALABLE MATHEMATICAL METHODS AND ALGORITHMS FOR EXTREME SCALE COMPUTING
- TSVIATKO RANGELOV
  Institute of Mathematics and Informatics,
  Bulgarian Academy of Science, Sofia, Bulgaria
  WAVE SCATTERING BY HETEROGENEITIES IN PIEZOELECTRIC PLANE VIA BOUNDARY INTEGRAL EQUATION METHOD
- BLAGOVEST SENDOV
  Institute of Mathematics and Informatics,
  Bulgarian Academy of Science, Sofia, Bulgaria
  ON THE GEOMETRY OF POLYNOMIALS
- ALEXANDAR YANOVSKI
  Department of Mathematics and Applied Mathematics,
  University of Cape Town, South Africa
  GAUGE COVARIANT THEORY OF THE GENERATING OPERATORS ASSOCIATED WITH LINEAR PROBLEMS OF CAUDREY-BEALS COIFMAN TYPE IN CANONICAL AND IN POLE GAUGE WITH AND WITHOUT REDUCTIONS

The present volume contains proceedings of the conference talks: plenary lectures and contributed talks (Part A), list of abstracts (Part B) and list of participants (Part C).

Angela Slavova
Chair of BGSIAM Section

Sofia, December 2014

CONTENTS

Second Order Semilinear Impulsive Differential Equations with Nonlocal Conditions

Haydar Akça, Valéry Covachev, Zlatinka Covacheva

Abstract

An abstract second order differential equation such that the linear part of the right-hand side is given by the infinitesimal generator of a strongly continuous cosine family of bounded linear operators while the nonlinear part satisfies a global Lipschitz condition, and provided with impulse and nonlocal conditions is studied. Theorems for existence and uniqueness of a mild and classical solution of the problem considered are proved.

AMS Subject Classification: 34A37, 34G20.
Key Words: impulse effect, nonlocal condition, cosine family.

1 Introduction

Many evolutionary processes in nature are characterized by the fact that at certain instants of time they experience a rapid change of their states. The theory of the impulsive differential equations is one of the attractive branches of differential equations which has extensive realistic mathematical modelling applications in physics, chemistry, engineering, and biological and medical sciences. The nonlocal condition generalizes the classical initial condition. In our previous papers [2, 3] we found sufficient conditions for the existence, uniqueness and continuous dependence of a mild solution of a first order impulsive functional-differential evolution nonlocal Cauchy problem such that the linear part part of the right-hand side of the differential equation is given by the infinitesimal generator of a strongly continuous semigroup of bounded linear operators.
In the present paper we consider the abstract second order nonlinear impulsive differential equation with nonlocal condition

$$\begin{aligned} x''(t) &= Ax(t) + f(t, x(t), x'(t)), \quad t \in (0,T] \setminus \{t_1, t_2, \dots, t_m\}, && (1) \\ \Delta x(t_k) &= I_k(x(t_k)), && \\ \Delta x'(t_k) &= \bar{I}_k(x(t_k), x'(t_k)), \quad k = \overline{1,m}, && (2) \\ x(0) &= x_0, && \\ x'(0) &= x_1 - g(x), && (3) \end{aligned}$$

where A is a linear operator from a real Banach space X with norm $\|\cdot\|$ into itself, $x: [0,T] \to X$, $\Delta x(t_k) = x(t_k+0) - x(t_k-0) \equiv x(t_k+0) - x(t_k)$, $\Delta x'(t_k) = x'(t_k+0) - x'(t_k-0) \equiv x'(t_k+0) - x'(t_k)$, $0 < t_1 < t_2 < \cdots < t_m < T$ are the instants of impulse effect, $f: [0,T] \times X^2 \to X$, $I_k: X \to X$, $\bar{I}_k: X^2 \to X$, $x_0, x_1 \in X$, and $g(x)$ is a function with values in X to be specified later.
In the present paper we present sufficient conditions for the existence and uniqueness of a mild and classical solution of problem (1)–(3). Our results generalize those of L. Byszewski and T. Winiarska [5] where equation (1) is provided with a nonlocal condition but without impulse effect. In our paper [1] a problem similar to (1)–(3) was formulated but not properly treated.
We make the assumptions:

A1 The operator A is the infinitesimal generator of a strongly continuous cosine family $\{C(t): t \in \mathbb{R}\}$ of bounded linear operators from X to itself.

A2 The adjoint operator A^* is densely defined in X^*, *i.e.*, $\overline{\mathcal{D}(A^*)} = X^*$.

Recall [5, 6] that the infinitesimal generator of a strongly continuous cosine family $\{C(t)\}$ is the operator $A: X \supset \mathcal{D}(A) \to X$ defined by

$$Ax := \frac{d^2}{dt^2} C(t)x \Big|_{t=0}, \quad x \in \mathcal{D}(A),$$

where

$$\mathcal{D}(A) := \left\{ x \in X : \ C(t)x \ \text{ is of class } \ C^2 \ \text{ with respect to } \ t \right\}.$$

Let us denote

$$E := \left\{ x \in X : \ C(t)x \ \text{ is of class } \ C^1 \ \text{ with respect to } \ t \right\}.$$

The associated sine family $\{S(t): t \in \mathbb{R}\}$ is defined by

$$S(t)x := \int_0^t C(s)x\, ds, \qquad x \in X, \quad t \in \mathbb{R}.$$

From Assumption **A1** it follows that [6] there are constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|C(t)\| \leq Me^{\omega|t|} \quad \text{and} \quad \|S(t)\| \leq Me^{\omega|t|} \quad \text{for} \quad t \in \mathbb{R}.$$

In particular, we have

$$\|C(t)\| \leq C \quad \text{and} \quad \|S(t)\| \leq C \quad \text{for} \quad t \in [0,T], \tag{4}$$

where $C = Me^{\omega T}$.
Following [5], we present a result obtained by J. Bochenek in [4].
Let us consider the Cauchy problem

$$\begin{aligned} x''(t) &= Ax(t) + h(t), \qquad t \in (0,T], \\ & x(0) = x_0, \quad x'(0) = x_1. \end{aligned} \tag{5}$$

Definition 1 A function $x: [0,T] \to X$ is said to be a *classical solution* of problem (5) if

$$
\begin{aligned}
& x \in C^1([0,T],X) \cap C^2((0,T],X), \\
& x(0) = x_0 \quad \text{and} \quad x'(0) = x_1, \\
& x''(t) = Ax(t) + h(t) \quad \text{for} \quad t \in (0,T].
\end{aligned}
$$

Theorem 1 *Suppose that*
(i) *Assumptions* **A1** *and* **A2** *are satisfied;*
(ii) $h: [0,T] \to X$ *is Lipschitz continuous;*
(iii) $x_0 \in \mathcal{D}(A)$ *and* $x_1 \in E$.
Then problem (5) *has a unique classical solution given by the formula*

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)h(s)\,ds, \quad t \in [0,T].$$

It is easy to see that this result can be generalized for the impulsive system

$$
\begin{aligned}
x''(t) &= Ax(t) + h(t), \quad t \in (0,T] \setminus \{t_1, t_2, \ldots, t_m\}, && (6)\\
\Delta x(t_k) &= I_k, \\
\Delta x'(t_k) &= \bar{I}_k, \quad k = \overline{1,m}, && (7)\\
x(0) &= x_0, \quad x'(0) = x_1. && (8)
\end{aligned}
$$

For convenience we denote $J = [0,T]$, $J_0 = [0,t_1]$, $J_k = (t_k, t_{k+1}]$, $k = \overline{1,m-1}$, $J_m = (t_m, T]$, $J' = J \setminus \{0, t_1, t_2, \ldots, t_m\}$. For a function $x: J \to X$ we denote by x_k the restriction of x to J_k, $k = \overline{0,m}$, with $\|x_k\|_{J_k} = \sup_{s \in J_k} \|x_k(s)\|$. If the function x_k happens to be differentiable, we denote its derivative by x_k'. Further we define the following classes of functions:
$PC(J,X) = \{x: J \to X \mid x_k \in C(J_k, X),\ k = \overline{0,m},$ and there exist $x(t_k+0),\ x(t_k-0),\ k = \overline{1,m},$ with $x(t_k) = x(t_k-0)\}$.
$PC^1(J,X) = \{x \in PC(J,X) \mid x_k' \in C(J_k,X),\ k = \overline{0,m},$ and there exist $x'(t_k+0),\ x'(t_k-0),\ k = \overline{1,m},$ with $x'(t_k) = x'(t_k-0)\}$.
$PC(J,X)$ is a Banach space with norm $\|x\|_{PC} = \max\{\|x_k\|_{J_k},\ k = \overline{0,m}\}$, and $PC^1(J,X)$ is a Banach space with norm $\|x\|_{PC^1} = \|x\|_{PC} + \|x'\|_{PC}$.

Definition 2 A function $x \in PC^1(J,X) \cap C^2(J',X)$ is called a *classical solution* of problem (6)–(8) if it satisfies the differential equation (6) on J' together with the impulse conditions (7) and the initial conditions (8).

Theorem 2 *Suppose that*
(i) *Assumptions* **A1** *and* **A2** *are satisfied;*
(ii) $h \in PC(J,X)$ *is such that its restrictions to* J_k *are Lipschitz continuous,* $k = \overline{0,m}$*;*
(iii) $x_0 \in \mathcal{D}(A)$ *and* $x_1 \in E$.

(iv) $I_k \in \mathcal{D}(A)$ *and* $\bar{I}_k \in E$ *for* $k = \overline{1,m}$.
Then problem (6)–(8) *has a unique classical solution given by the formula*

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)h(s)\,ds \tag{9}$$
$$+ \sum_{0<t_k<t} C(t-t_k)I_k + \sum_{0<t_k<t} S(t-t_k)\bar{I}_k, \quad t \in [0,T].$$

Theorem 2 can be proved by applying Theorem 1 on each interval of continuity J_k, $k = \overline{0,m}$.
This theorem suggests the following definition.

Definition 3 A function $x \in PC^1(J,X)$ satisfying the integro-summary equation

$$x(t) = C(t)x_0 + S(t)(x_1 - g(x)) + \int_0^t S(t-s)f(s,x(s),x'(s))\,ds \tag{10}$$
$$+ \sum_{0<t_k<t} C(t-t_k)I_k(x(t_k)) + \sum_{0<t_k<t} S(t-t_k)\bar{I}_k(x(t_k),x'(t_k)), \quad t \in [0,T].$$

is said to be a *mild solution* of the nonlocal problem (1)–(3).

2 Main results

Theorem 3 *Suppose that*
(i) *Assumption* **A1** *is satisfied;*
(ii) *The function* $t \mapsto f(t,x,y)$ *belongs to* $PC(J,X)$ *and there exists a positive constant* L_1 *such that*

$$\|f(t,x,y) - f(t,\tilde{x},\tilde{y})\| \le L_1(\|x-\tilde{x}\| + \|y-\tilde{y}\|)$$

for $t \in [0,t]$, $x,\tilde{x},y,\tilde{y} \in X$;
(iii) $g:\ PC^1(J,X) \to X$ *and there exists a positive constant* L_2 *such that*

$$\|g(x) - g(\tilde{x})\| \le L_2\|x-\tilde{x}\|_{PC^1}.$$

for $x,\tilde{x} \in PC^1(J,X)$;
(iv) $I_k:\ X \to E$ *and* $\bar{I}_k:\ X^2 \to X$ *and there exist positive constants* L_3 *and* L_4 *such that*

$$\|I_k(x) - I_k(\tilde{x})\| \le L_3\|x-\tilde{x}\| \quad \text{and} \quad \|\bar{I}_k(x,y) - \bar{I}_k(\tilde{x},\tilde{y})\| \le L_4(\|x-\tilde{x}\| + \|y-\tilde{y}\|)$$

for $k = \overline{1,m}$, $x,\tilde{x},y,\tilde{y} \in X$;
(v) $\tilde{C}(TL_1 + L_2 + mL_3 + mL_4) < 1$, *where* $\tilde{C} = \max\{C, C'\}$, C *was defined in* (4) *and* $C' = \sup\{\|C'(t)\|:\ t \in [0,t]\}$;
(vi) $x_0 \in E$ *and* $x_1 \in X$.
Then problem (1)–(3) *has a unique mild solution.*

Remark 1 We call equation (1) *semilinear* since its nonlinearities satisfy global Lipschitz conditions with sufficiently small constants.

Proof. We can write equation (10) in an operator form

$$x = \mathcal{F}x$$

where the operator $\mathcal{F}: \ PC^1(J,X) \to PC^1(J,X)$ is defined by

$$\begin{aligned} &(\mathcal{F}u)(t) = C(t)x_0 + S(t)(x_1 - g(u)) + \int_0^t S(t-s) f(s,u(s),u'(s))\, ds \\ + &\sum_{0<t_k<t} C(t-t_k) I_k(u(t_k)) + \sum_{0<t_k<t} S(t-t_k)\bar{I}_k(u(t_k),u'(t_k)), \quad t \in [0,T]. \end{aligned}$$

Now we show that $\mathcal{F}$ is a contraction on the Banach space $PC^1(J,X)$. In fact, for $u, \tilde{u} \in PC^1(J,X)$ we have

$$\begin{aligned} &(\mathcal{F}u)(t) - (\mathcal{F}\tilde{u})(t) = -S(t)(g(u) - g(\tilde{u})) \\ + &\int_0^t S(t-s)\left(f(s,u(s),u'(s)) - f(s,\tilde{u}(s),\tilde{u}'(s))\right) ds \\ + &\sum_{0<t_k<t} C(t-t_k)\left(I_k(u(t_k)) - I_k(\tilde{u}(t_k))\right) \\ + &\sum_{0<t_k<t} S(t-t_k)\left(\bar{I}_k(u(t_k),u'(t_k)) - \bar{I}_k(\tilde{u}(t_k),\tilde{u}'(t_k))\right), \quad t \in [0,T], \end{aligned}$$

hence

$$\begin{aligned} &\|(\mathcal{F}u)(t) - (\mathcal{F}\tilde{u})(t)\| \le \|S(t)\| \cdot \|g(u) - g(\tilde{u})\| \\ + &\int_0^t \|S(t-s)\| \cdot \|f(s,u(s),u'(s)) - f(s,\tilde{u}(s),\tilde{u}'(s))\|\, ds \\ + &\sum_{0<t_k<t} \|C(t-t_k)\| \cdot \|I_k(u(t_k)) - I_k(\tilde{u}(t_k))\| \\ + &\sum_{0<t_k<t} \|S(t-t_k)\| \cdot \|\bar{I}_k(u(t_k),u'(t_k)) - \bar{I}_k(\tilde{u}(t_k),\tilde{u}'(t_k))\| \\ \le &\ CL_2\|u - \tilde{u}\|_{PC^1} + C\int_0^t L_1\left(\|u(s) - \tilde{u}(s)\| + \|u'(s) - \tilde{u}'(s)\|\right) ds \\ + &\ C\sum_{0<t_k<t} \left\{L_3\|u(t_k) - \tilde{u}(t_k)\| + L_4\left(\|u(t_k) - \tilde{u}(t_k)\| + \|u'(t_k) - \tilde{u}'(t_k)\|\right)\right\} \\ \le &\ C\left\{mL_3\|u - \tilde{u}\|_{PC} + (TL_1 + L_2 + mL_4)\|u - \tilde{u}\|_{PC^1}\right\}. \qquad (11) \end{aligned}$$

Similarly we obtain

$$
\begin{aligned}
&(\mathcal{F}u)'(t) - (\mathcal{F}\tilde{u})'(t) = -C(t)(g(u) - g(\tilde{u})) \\
+ \quad & \int_0^t C(t-s)\left(f(s,u(s),u'(s)) - f(s,\tilde{u}(s),\tilde{u}'(s))\right) ds \\
+ \quad & \sum_{0<t_k<t} C'(t-t_k)\left(I_k(u(t_k)) - I_k(\tilde{u}(t_k))\right) \\
+ \quad & \sum_{0<t_k<t} C(t-t_k)\left(\bar{I}_k(u(t_k),u'(t_k)) - \bar{I}_k(\tilde{u}(t_k),\tilde{u}'(t_k))\right), \quad t \in [0,T],
\end{aligned}
$$

hence

$$
\|(\mathcal{F}u)'(t) - (\mathcal{F}\tilde{u})'(t)\| \le C' m L_3 \|u-\tilde{u}\|_{PC} + C\left(TL_1 + L_2 + mL_4\right)\|u-\tilde{u}\|_{PC^1}. \quad (12)
$$

From (11) and (12) we derive the estimate

$$
\begin{aligned}
& \|\mathcal{F}u - \mathcal{F}\tilde{u}\|_{PC^1} \\
\le \quad & m(C+C')L_3\|u-\tilde{u}\|_{PC} + 2C\left(TL_1 + L_2 + mL_4\right)\|u-\tilde{u}\|_{PC^1} \\
\le \quad & 2\tilde{C}(TL_1 + L_2 + mL_3 + mL_4)\|u-\tilde{u}\|_{PC^1} < \|u-\tilde{u}\|_{PC^1}.
\end{aligned}
$$

Thus the contraction mapping $\mathcal{F}$ has a unique fixed point $x \in PC^1(J,X)$, which is the mild solution of problem (1)–(3). □

Similarly to Definition 2, we give the following definition.

Definition 4 A function $x \in PC^1(J,X) \cap C^2(J',X)$ is called a *classical solution* of problem (1)–(3) if it satisfies the differential equation (1) on J' together with the impulse conditions (2) and the nonlocal initial conditions (3).

Theorem 4 *Suppose that*
(i) *Assumptions* **A1** *and* **A2** *are satisfied;*
(ii) *The function $t \mapsto f(t,x,y)$ belongs to $PC(J,X)$ and there exists a positive constant $\tilde{L}_1$ such that*

$$
\|f(t,x,y) - f(\tilde{t},\tilde{x},\tilde{y})\| \le \tilde{L}_1(|t-\tilde{t}| + \|x-\tilde{x}\| + \|y-\tilde{y}\|)
$$

for $t,\tilde{t} \in J_k,\ k = \overline{0,m},\ x,\tilde{x},y,\tilde{y} \in X$;
(iii) *$g:\ PC^1(J,X) \to E$ and there exists a positive constant L_2 such that*

$$
\|g(x) - g(\tilde{x})\| \le L_2\|x-\tilde{x}\|_{PC^1}.
$$

for $x,\tilde{x} \in PC^1(J,X)$;
(iv) *$I_k:\ X \to \mathcal{D}(A)$ and $\bar{I}_k:\ X^2 \to E$ and there exist positive constants L_3 and L_4 such that*

$$
\|I_k(x) - I_k(\tilde{x})\| \le L_3\|x-\tilde{x}\| \quad \text{and} \quad \|\bar{I}_k(x,y) - \bar{I}_k(\tilde{x},\tilde{y})\| \le L_4(\|x-\tilde{x}\| + \|y-\tilde{y}\|)
$$

for $k = \overline{1,m}$, $x, \tilde{x}, y, \tilde{y} \in X$*;*
(v) $\tilde{C}(T\tilde{L}_1 + L_2 + mL_3 + mL_4) < 1$*;*
(vi) $x_0 \in \mathcal{D}(A)$ *and* $x_1 \in E$*.*
Then problem (1)–(3) *has a unique classical solution.*

Proof. Since all assumptions of Theorem 3 are satisfied, problem (1)–(3) has a unique mild solution x. We shall show that x is a classical solution of problem (1)–(3). First we show that x and x' are Lipschitz continuous on each interval J_k, $k = \overline{0,m}$. More precisely, we show that there exist positive constants C_k such that

$$\|x(\tilde{t}) - x(t)\| + \|x'(\tilde{t}) - x'(t)\| \leq C_k|\tilde{t} - t| \quad \text{for} \quad t, \tilde{t} \in J_k, \quad k = \overline{0,m}. \tag{13}$$

We use induction on k. First suppose that $0 < t < t + h \leq t_1$. Then we have

$$x(t+h) - x(t) = \varphi_0(t+h) - \varphi_0(t)$$
$$+ \int_0^{t+h} S(t+h-s)f(s, x(s), x'(s))\, ds - \int_0^t S(t-s)f(s, x(s), x'(s))\, ds,$$

where by our assumptions the function

$$\varphi_0(t) = C(t)x_0 + S(t)(x_1 - g(x))$$

is of class $C^2(J_0)$, thus there exist constants C_{10} and C_{20} such that

$$\|\varphi_0(t+h) - \varphi_0(t)\| \leq C_{10}h \quad \text{and} \quad \|\varphi_0'(t+h) - \varphi_0'(t)\| \leq C_{20}h.$$

Thus

$$\begin{aligned}
\|x(t+h) - x(t)\| \;&\leq\; C_{10}h + \left\| \int_0^h S(t+h-s)f(s, x(s), x'(s))\, ds \right\| \\
+ &\left\| \int_h^{t+h} S(t+h-s)f(s, x(s), x'(s))\, ds - \int_0^t S(t-s)f(s, x(s), x'(s))\, ds \right\| \\
\leq &\;(C_{10} + CK)h + \left\| \int_0^t S(t-s)\big(f(s+h, x(s+h), x'(s+h)) - f(s, x(s), x'(s))\big)\, ds \right\| \\
\leq &\;(C_{10} + CK)h + \int_0^t C\tilde{L}_1\, (h + \|x(s+h) - x(s)\| + \|x'(s+h) - x'(s)\|)\, ds,
\end{aligned}$$

where

$$K = \sup\{\|f(t, x(t), x'(t))\| :\ t \in [0,T]\}.$$

We can write the above inequality in the form

$$\|x(t+h) - x(t)\| \;\leq\; C_{30}h \tag{14}$$
$$+ \quad C_{40}\int_0^t (\|x(s+h) - x(s)\| + \|x'(s+h) - x'(s)\|)\, ds,$$

where C_{30} and C_{40} are some positive constants.
Further on, we have

$$x'(t) = \varphi_0'(t) + \int_0^t C(t-s)f(s, x(s), x'(s))\, ds,$$

and in a similar way we obtain

$$\begin{aligned} \|x'(t+h) - x'(t)\| &\le C_{50}h \\ + \quad C_{60} \int_0^t & (\|x(s+h) - x(s)\| + \|x'(s+h) - x'(s)\|)\, ds, \end{aligned} \tag{15}$$

Adding together (14) and (15), we obtain the inequality

$$\begin{aligned} \|x(t+h) - x(t)\| + \|x'(t+h) - x'(t)\| &\le (C_{30} + C_{50})h \\ + \quad (C_{40} + C_{60}) \int_0^t & (\|x(s+h) - x(s)\| + \|x'(s+h) - x'(s)\|)\, ds. \end{aligned}$$

Using Gronwall's inequality, we obtain

$$\|x(t+h) - x(t)\| + \|x'(t+h) - x'(t)\| \le C_0 h$$

which in fact is inequality (13) for the interval J_0.
In the inductive step we suppose that inequality (13) is satisfied on all intervals J_i, $i = \overline{0, k-1}$, and we show that it is also valid on the interval J_k. For brevity we omit the details.
Thus for $t, \tilde{t} \in J_k$ we have

$$\begin{aligned} & \|f(\tilde{t}, x(\tilde{t}), x'(\tilde{t})) - f(t, x(t), x'(t))\| \\ \le \quad & \tilde{L}_1\big(|\tilde{t} - t| + \|x(\tilde{t}) - x(t)\| + \|x'(\tilde{t}) - x'(t)\|\big) \ \le\ \tilde{L}_1(1 + C_k)|\tilde{t} - t|, \end{aligned}$$

which shows that the restrictions of the mapping $[0, T] \ni t \mapsto f(t, x(t), x'(t)) \in X$ to each interval J_k are Lipschitz continuous.
This property and the assumptions of Theorem 4 allow us to apply Theorem 2 to the linear impulsive system

$$\begin{aligned} u''(t) &= Au(t) + f(t, x(t), x'(t)), \quad t \in (0, T] \setminus \{t_1, t_2, \ldots, t_m\}, \\ \Delta u(t_k) &= I_k(x(t_k)), \\ \Delta u'(t_k) &= \bar{I}_k(x(t_k), x'(t_k)), \quad k = \overline{1, m}, \\ u(0) &= x_0, \\ u'(0) &= x_1 - g(x). \end{aligned}$$

This system has a unique classical solution u given by the formula

$$\begin{aligned} u(t) = C(t)x_0 + S(t)(x_1 - g(x)) + \int_0^t S(t-s)f(s, x(s), x'(s))\, ds \\ + \quad \sum_{0<t_k<t} C(t-t_k)I_k(x(t_k)) + \sum_{0<t_k<t} S(t-t_k)\bar{I}_k(x(t_k), x'(t_k)), \quad t \in [0, T]. \end{aligned}$$

Since the unique mild solution x of system (1)–(3) satisfies equality (10), this implies that x is the unique classical solution of this system. □

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